

# Exact solution of the one-dimensional ballistic aggregation

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## Abstract

An exact expression for the mass distribution  $\rho(M, t)$  of the ballistic aggregation model in one dimension is derived in the long time regime. It is shown that it obeys scaling  $\rho(M, t) = t^{-4/3} F(M/t^{2/3})$  with a scaling function  $F(z) \sim z^{-1/2}$  for  $z \ll 1$  and  $F(z) \sim \exp(-z^3/12)$  for  $z \gg 1$ . Relevance of these results to Burgers turbulence is discussed.

Ballistic aggregation provides a simple model of nonequilibrium statistical physics which is a natural version of a dissipative gas of hard spheres where particles follow the basic laws of mechanics. It consists in a one-dimensional gas of point-like massive particles which move freely until they collide. The perfectly inelastic collision of two masses conserves the total mass and momentum, while dissipation occurs as kinetic energy is lost in each collision. One can anticipate the formation of more and more massive while slower and slower aggregates.

This model was introduced by Carnevale, Pomeau and Young [1] where they conjectured, based on scaling arguments and numerical simulations, an asymptotic scaling regime for the mass distribution  $\rho(M, t) = F(M/\langle M \rangle_t)/\langle M \rangle_t^2$ . The average mass per aggregate was supposed to grow algebraically with time as  $\langle M \rangle_t \sim t^{2/3}$  and the scaling function had a simple universal exponential form  $F(z) = \exp(-z)$  independent of the initial conditions. Later, this conjecture was reinforced by Piasecki [2] where he solved the hierarchy of dynamical

equations governing the system inside a mean-field approximation scheme.

This system, in its continuous limit, was also studied as a simplified astronomical model for the agglomeration of cosmic dust into macroscopic objects [3]. In the ballistic aggregation model, the aggregates interact only through their collisions. An aggregation model where gravitational interactions are present has been studied in [4].

It is important to mention the connection between this model and some solutions of the Burgers equation. At very high Reynolds number, the asymptotic solution of the Burgers equation consists of a train of shock waves. The laws of motion which govern the dynamics of these shock waves are found to be equivalent to a ballistic aggregation system (see [5]).

In this letter, I verify the scaling hypothesis for the mass distribution and find in an exact calculation an explicit form for the scaling function. It happens to be different from the conjectured simple exponential.

Rather than solving the set of partial differential equations governing the evolution of the system, I exploit the fact that, once the initial state of the system is given, the dynamics is completely deterministic. Our approach will thus be based on a statistical study of the initial conditions and is largely inspired by the work of Martin and Piasecki [6].

Initially, particles having all the same mass  $m$  are regularly placed on a line with the same inter-particle distance  $a$ . Initial mass density is thus  $\rho_0 = m/a$ . The initial momentum of the thermalized particles are not correlated and are distributed according to the same Gaussian distribution  $\phi(p) = \sqrt{\beta/(2\pi m)} \exp(-\beta p^2/(2m))$  where I now choose  $\beta = 1/2$  without loss of generality.

I compute now the density distribution  $\rho_m(X, M, P, t)$  where  $\rho_m(X, M, P, t)dM dP dX$  is the number of aggregates located in  $(X, X + dX)$  with momentum in  $(P, P + dP)$  and mass in  $(M, M + dM)$  at time  $t$ .

When the coordinates  $(X, M, P, t)$  of an aggregate are given, they uniquely define the number  $n = M/m$  as well as the initial positions  $X - Pt/M - M/(2\rho_0) \leq x_i \leq X - Pt/M + M/(2\rho_0)$  ( $i = 1, \dots, n$ ) of its constituents. A crucial point is that an aggregate, once formed, is moving according to the movement of the center of mass (CM) of its constituents, which

can be determined from the initial state. I label the location of the CM at time  $t$  of the  $r$  particles located initially at  $(j+1)a, (j+2)a, \dots, (j+r)a$  by

$$X_{j+1}^r(t) := \frac{1}{rm} \sum_{i=1}^r m x_{j+i} + t p_{j+i} = j a + \frac{r+1}{2} a + \frac{t}{rm} \sum_{i=1}^r p_{j+i}. \quad (1)$$

The mass distribution can thus be determined from the initial conditions and an aggregate of mass  $M = mn$  is present at time  $t$  iff

- the CM of its leftmost  $s$  particles has met the CM of its rightmost  $n-s$  particles for all  $s = 1, \dots, n-1$  up to time  $t$ , leading to  $X_{j+1}^s(t) > X_{j+s+1}^{n-s}(t)$  for  $1 \leq s \leq n-1$ ,
- the CM of the successive groups of particles not constituting the aggregate has not met the CM of the aggregate, thus  $X_{j-r+1}^r(t) < X < X_{j+n+1}^r(t)$  for  $r \geq 1$ .

One has (see [6] for details)

$$\begin{aligned} \rho_m(X, M, P, t) = & \left\langle \prod_{r=1}^{\infty} \theta \{X - X_{j-r+1}^r(t)\} \theta \{X_{j+r+1}^r(t) - X\} \right. \\ & \times \left. \prod_{s=1}^{n-1} \theta \{X_{j+1}^s(t) - X_{j+n+1}^{n-s}(t)\} \delta \left( P - \sum_{r=1}^n p_{j+r} \right) \right\rangle \end{aligned} \quad (2)$$

with  $\theta$  the Heavyside step function and where  $M = nm$  and  $X = (2j + n + 1)a/2 + tP/M$ . The brackets denote the average over the initial distribution of the momentum.

Using Eq.(1) and the Gaussian form of the initial distribution, one finds the exact scaling form

$$\rho_m(X, M, P; t) = \rho_m(M, P; t) = \frac{1}{t^{1/3}} \rho_{m'}(M', P') \quad (3)$$

with  $M' = M/t^{2/3}$ ,  $P' = P/t^{1/3}$  and  $m' = m/t^{2/3}$ . Note that, due to translational invariance, the mass distribution does not depend on  $X$ .

Owing to the uncorrelated initial Gaussian distribution of the momentum, one can compute the density  $\rho$  using an analogy with a Brownian motion in the momentum space under particular constraints [6,7], (see Fig.(1)). One finds

$$\rho_{m'}(M', P') = J_{m'} \left( -M' - \frac{P'}{M'} \right) I_{m'}(M', P') J_{m'} \left( -M' + \frac{P'}{M'} \right) \quad (4)$$

where  $J_{m'}(Z)$  is the probability for a Brownian motion  $P(\tau)$  to start from  $P(0) = 0$  and pass above the discrete points  $P(rm') > Zrm' - (rm')^2$  ( $r \geq 1$ ), and  $I_{m'}(M', P')$  is the probability for a Brownian motion to start at  $P(0) = 0$ , end at  $P(M') = P'$  and over-passing the discrete points  $P(rm') > (M' + P'/M')rm' - (rm')^2$  ( $1 \geq r \geq n$ ).

# FIGURES

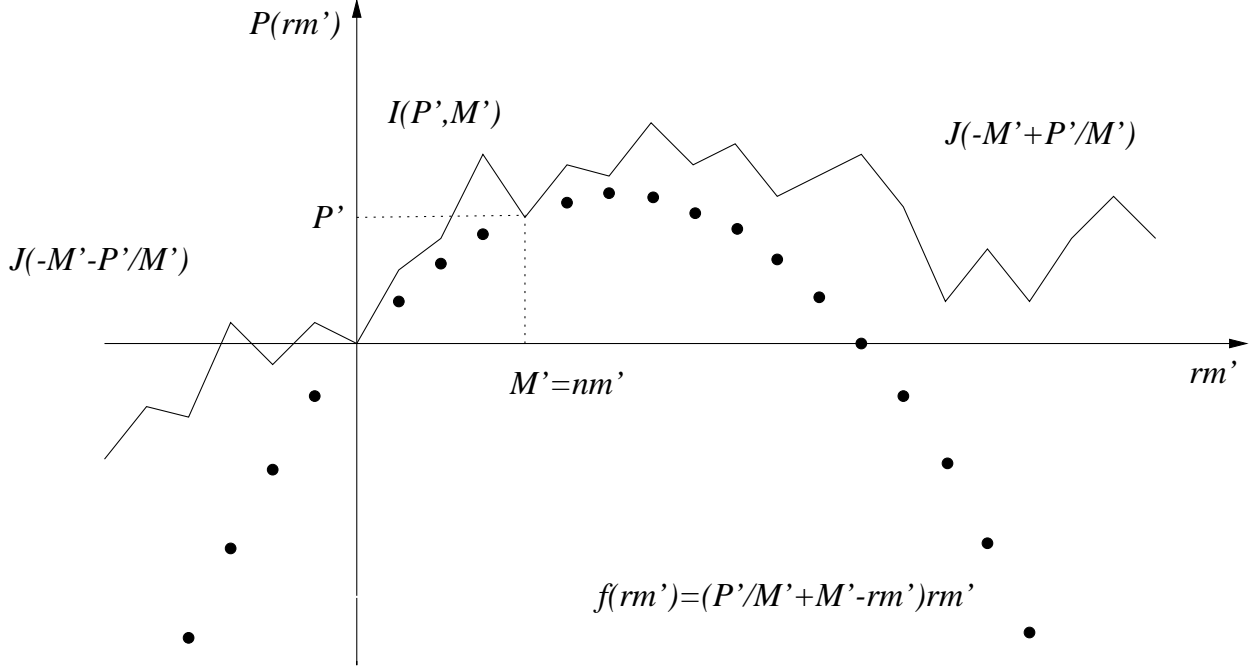


FIG. 1. The Brownian motion used in the construction of our solution.

I will derive below an expression for the mass distribution in the limit  $m' = m/t^{2/3} \rightarrow 0$  which is reached either when  $t \rightarrow \infty$  for a fixed  $m$  (asymptotic long time limit) or for any fixed time  $t$  when  $m \rightarrow 0$  (continuous limit). In this limit, one keeps  $M'$  and  $P'$  of order  $\mathcal{O}(1)$ . In terms of the Brownian motion introduced above, the space  $m'$  between the discrete points barrier shrinks to zero and approaches a continuous barrier which makes the problem tractable analytically. Nevertheless, the functions  $I$  and  $J$  are identically null for  $m' = 0$ . One should thus keep track of the first space  $m'$  (the Brownian motion will be unrestricted up to the first point of the barrier) and will find a mass distribution which is an expansion in power of  $m'$ . From now on, I drop the subscript  $'$  and set  $\rho_0 = 1/2$  without loss of generality.

One finds the dominant contribution in  $m$ :

$$\begin{aligned} \bar{I}_m(M, P) &= e^{-\frac{P^2}{2M}} \int_{Mm-m^2}^{\infty} dP_1 \phi(P_1) \int_{-Mm-m^2}^{\infty} dP_2 \phi(P_2) K_M(P_1, m, P_2, M-m) \\ &= \frac{m}{\pi} e^{-\frac{P^2}{2M}} \left. \frac{\partial^2}{\partial P_1 \partial P_2} K_M(m, P_1, M-m, P_2) \right|_{P_1=P_2=0} + \mathcal{O}(m^2) \end{aligned} \quad (5)$$

and

$$\begin{aligned}\bar{J}_m(Y) &= \int_{Ym-m^2}^{\infty} dP_1 \phi(P_1) \lim_{N \rightarrow \infty} \int_{YNm-(Nm)^2}^{\infty} dP_2 K_Y(P_1, m, P_2, Nm) \\ &= \sqrt{\frac{m}{\pi}} \lim_{N \rightarrow \infty} \int_{YNm-(Nm)^2}^{\infty} dP_2 \left. \frac{\partial}{\partial P_1} K_Y(P_1, m, P_2, Nm) \right|_{P_1=0} + \mathcal{O}(m^{3/2})\end{aligned}\quad (6)$$

where  $K_Z(P_1, \tau_1, P_2, \tau_2)$  is the probability for a Brownian motion to start at  $P(\tau_1) = P_1$ , end at  $P(\tau_2) = P_2$  while staying above the continuous barrier  $P(\tau) > f(\tau) = Z\tau - \tau^2$ .

Defining the stochastic process  $Q(\tau) = P(\tau) - f(\tau)$ , one has [6]

$$K_Z(P_1, \tau_1, P_2, \tau_2) = G(Q_1, \tau_1, Q_2, \tau_2) \exp\left(\frac{1}{2}(Q_1 f'(\tau_1) - Q_2 f'(\tau_2) - \frac{1}{2} \int_{\tau_1}^{\tau_2} d\tau f'(\tau)^2)\right) \quad (7)$$

where  $Q_i = P_i - f(\tau_i)$ , ( $i = 1, 2$ ). The function  $G$  is the solution of the equation [6]

$$\left(\frac{\partial}{\partial \tau_2} - \frac{\partial^2}{\partial Q_2^2} - \frac{Q_2}{2} f''(\tau_2)\right) G(Q_1, \tau_1, Q_2, \tau_2) = 0 \quad (8)$$

with  $G(Q_1, \tau, Q_2, \tau) = \delta(Q_1 - Q_2)$  and  $G(0, \tau_1, Q_2, \tau_2) = G(Q_1, \tau_1, 0, \tau_2) = 0$ . In our problem  $f''(\tau) = 2$ .

The equation (8) can be solved (see [7] for details) and one finds

$$G(Q_1, \tau_1, Q_2, \tau_2) = \sum_{k \geq 1} e^{-\omega_k(\tau_2 - \tau_1)} \frac{\text{Ai}(Q_1 - \omega_k) \text{Ai}(Q_2 - \omega_k)}{(\text{Ai}'(-\omega_k))^2} \quad (9)$$

where  $\text{Ai}$  is the Airy function [8] which has an infinite countable numbers of zeroes  $-\omega_k$  on the negative real axe ( $-\omega_1 = -2.33811 \dots, -\omega_2 = -4.08795 \dots, \dots$ ). This function had to be expected in this problem as it is known that it arises in the description of a Brownian motion with a parabolic drift [9].

Using Eqs.(5,7,9), one gets

$$\bar{I}_m(M) = \frac{m}{\pi} e^{-M^3/12} \mathcal{I}(M) + \mathcal{O}(m^2) \quad (10)$$

with

$$\mathcal{I}(M) = \sum_{k \geq 1} e^{-\omega_k M}. \quad (11)$$

In the same way, I use Eqs.(6,7,9) and obtain

$$\bar{J}_m(Y) = \sqrt{\frac{m}{\pi}} \lim_{\mathcal{M} \rightarrow \infty} e^{(Y/2 - \mathcal{M})^3/3 - (Y/2)^3/3} \int_0^{\infty} dx e^{-x(Y/2 - \mathcal{M})} \sum_{k \geq 1} e^{-\omega_k \mathcal{M}} \frac{\text{Ai}(x - \omega_k)}{\text{Ai}'(-\omega_k)} + \mathcal{O}(m^{3/2}). \quad (12)$$

Using the integral representation of the sum in (12) [9],

$$\sum_{k \geq 1} e^{-\omega_k \mathcal{M}} \frac{\text{Ai}(x - \omega_k)}{\text{Ai}'(-\omega_k)} = \int_{c-i\infty}^{c+i\infty} dz e^{z\mathcal{M}} \frac{\text{Ai}(z+x)}{\text{Ai}(z)} \quad (13)$$

with  $c > -\omega_1$ , one can exchange the integration and find after a tedious analytical calculation [7]

$$\bar{J}_m(Y) = \sqrt{\frac{m}{\pi}} e^{-(Y/2)^3/3} \mathcal{J}(Y) + \mathcal{O}(m^{3/2}) \quad (14)$$

with

$$\mathcal{J}(Y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz \frac{e^{zY/2}}{\text{Ai}(z)} \quad (15)$$

where  $c > -\omega_1$ .

Now, inserting the expression for  $I$  and  $J$  in Eq.(4), one has in the original variables

$$\rho(M, P; t) = \frac{m^2}{t^{5/3}\pi^2} \mathcal{I}\left(\frac{M}{t^{2/3}}\right) \mathcal{J}\left(-\frac{M}{t^{2/3}} + \frac{Pt^{1/3}}{M}\right) \mathcal{J}\left(-\frac{M}{t^{2/3}} - \frac{Pt^{1/3}}{M}\right) + \imath \left(\frac{m^2}{t^{5/3}}\right) \quad (16)$$

From this equation one has that the concentration  $c(t)$  of aggregates, the aggregates average mass and momentum and the mean energy per unit of length  $E(t)$  behave, for time  $t \gg 1$ , as

$$c(t) \sim t^{-2/3}, \quad \langle M \rangle_t \sim t^{2/3}, \quad \langle P \rangle_t = 0, \quad \sqrt{\langle P^2 \rangle_t} \sim t^{1/3}, \quad E(t) \sim t^{-2/3}. \quad (17)$$

A careful integration over  $P$  [7] leads to the mass distribution, which is the main result of this letter,

$$\rho(M, t) = \frac{1}{t^{4/3}} F\left(\frac{M}{t^{2/3}}\right) + \imath \left(\frac{m^2}{t^{4/3}}\right) \quad (18)$$

where one sees that it obeys the expected scaling form with a scaling function

$$F(M') = 2 \frac{m^2}{\pi^2} M' \mathcal{I}(M') \mathcal{H}(M') \quad (19)$$

where

$$\mathcal{I}(M') = \sum_{k \geq 1} e^{-\omega_k M'}, \quad \mathcal{H}(M') = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz \frac{e^{-M'z}}{\text{Ai}^2(z)} \quad (20)$$

with  $c > -\omega_1$ .

The scaling function  $F(M')$  is plotted on Fig.2.

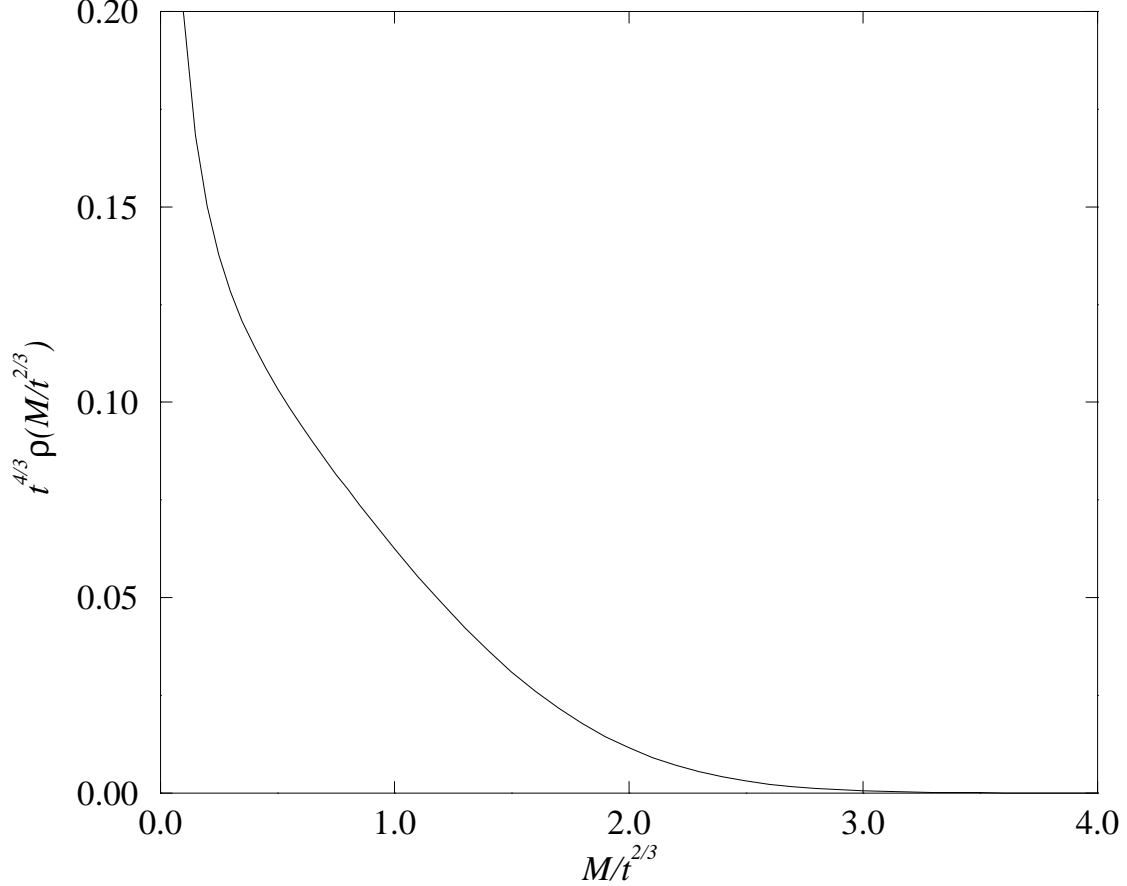


FIG. 2. The rescaled mass distribution  $t^{4/3}\rho(M,t)$  as a function of  $M/t^{2/3}$ .

One can compare the obtained scaling function with the conjectured one ( $F_{\text{conj.}}(M') = \exp(-M')$ ) [1]. In particular small and large arguments present strong differences. Indeed, for  $M' \ll 1$ , one get  $\mathcal{H}(M') = 1 + \mathcal{O}(M')$  while one can estimate  $\mathcal{I}(M')$  using the asymptotic properties of the zeroes of the Airy function  $\omega_k = [(3\pi k)/2]^{2/3} + \mathcal{O}(k^{-1/3})$  and find  $\mathcal{I}(M') \sim (2\sqrt{\pi}M'^{3/2})^{-1}$ . One thus find

$$F(M') = \frac{m^2}{\pi^{5/2}} \frac{1}{\sqrt{M'}} + \mathcal{O}(\sqrt{M'}), \quad (M' \ll 1). \quad (21)$$

One can conclude, for example, that the number  $N(M_0, t) = \int_0^{M_0} \rho(M, t) dM$  of aggregates



of small masses  $M < M_0 \ll t^{2/3}$  at time  $t$  is well underestimated by the conjectured form which leads to  $N(M_0, t) \sim M_0/t$ , while the exact solution gives  $N(M_0, t) \sim \sqrt{M_0}/t$ .

For  $M' \gg 1$ , one can estimate the function  $\mathcal{H}(M')$  by the steepest descent method and find  $\mathcal{H}(M') \sim \sqrt{\pi} M'^{3/2} \exp(-M'^3/12)$ . On the other end, one has  $\mathcal{I}(M') \sim \exp(-\omega_1 z)$  and finally

$$F(M') = \frac{2}{\pi^{3/2}} M'^{5/2} e^{-M' \omega_1 - M'^3/12}, \quad (M' \gg 1). \quad (22)$$

This is again different from the conjectured function as large masses  $M \gg t^{2/3}$  have a much smaller chance to be present in the system.

Notice that the asymptotic behaviors of the scaling function are compatible with the exact bounds found for the burgers problem [10] with white noise initial condition. On the other end, Eq.(18) solves the shock strength distribution questioned in [5] and studied numerically in [11].

It is instructive to compute, along the same line, the collision frequency  $\nu_2(M_1, M_2, t)$  where  $\nu_2 dM_1 dM_2 dt$  is the number of collision per unit of volume between masses in  $(M_1, M_1 + dM_1)$  and  $(M_2, M_2 + dM_2)$  in a time in  $(t, t + dt)$ . I find

$$\begin{aligned} \nu_2(M_1, M_2, t) = & \int_{-\infty}^{\infty} dP_1 \int_{-\infty}^{\infty} dP_2 \left| \frac{P_1}{M_1} - \frac{P_2}{M_2} \right| \delta \left( t \left( \frac{P_1}{M_1} - \frac{P_2}{M_2} \right) - M_1 - M_2 \right) \\ & \times t^{-2/3} J_{m'} \left( -\frac{P'_1}{M'_1} - M'_1 \right) I_{m'}(M'_1, P'_1) I_{m'}(M'_2, P'_2) J_{m'} \left( \frac{P'_2}{M'_2} - M'_2 \right), \end{aligned} \quad (23)$$

with  $M'_i = M_i/t^{2/3}$ ,  $P'_i = P_i/t^{1/3}$  and  $m' = m/t^{2/3}$ , leading to

$$\nu_2(M_1, M_2, t) \sim \left( \frac{m}{t\pi} \right)^3 \mathcal{F} \left( \frac{M_1}{t^{2/3}}, \frac{M_2}{t^{2/3}} \right) \quad (24)$$

with

$$\mathcal{F}(M'_1, M'_2) = (M'_1 + M'_2) M'_1 M'_2 \mathcal{I}(M'_1) \mathcal{I}(M'_2) \mathcal{H}(M'_1 + M'_2) \quad (25)$$

and  $\mathcal{I}$  and  $\mathcal{H}$  as above. This collision frequency clearly does not factorize in a product of functions of  $M_1$  and  $M_2$ , respectively. This fact invalidates the assumption on which the mass distribution was computed in [2].

One can inquire about the universality of these results with respect to other initial conditions. Let us first consider a Poissonian distribution of the particles initial positions with an average interparticle distance  $a$ . The discrete points over which the Brownian motion should pass in the construction of our solution are still distributed on the same parabola but with irregular spacing. In the long time limit and after rescaling, the spacing between points of average  $a' = a/t^{2/3}$  become smaller and smaller up to be, in first order in  $m'$ , a continuum. The difference between irregular and regular spacing is thus asymptotically erased and the result Eq.(18) should be recovered in this case.

A bimodal momentum distribution  $\phi(p) = (\delta(p - p_0) + \delta(p + p_0))/2$  is used in the initial state in [1]. I believe that this should not affect the form of the mass distribution (18) as the random walk initiated by this distribution is well approximated, in the long time limit, by the considered Brownian motion.

One can define a distribution where momentum are initially correlated. In this case, one expects the scaling function to be different, at least for small  $M'$  [12].

In summary, I found an exact asymptotic solution for the mass distribution of the ballistic aggregation in one dimension. Such an exact solution is not frequent in a nonequilibrium system and has permitted to verify scaling hypothesis for this system. While the average mass per aggregate was proved to behave with time as  $\langle M \rangle_t \sim t^{2/3}$  for  $t \gg 1$ , as expected from previous studies, the scaling function is shown here to be different from the conjectured one. This distribution also solve the shock strength distribution of the one dimensional Burgers equation in the inviscid limit with a white noise initial condition.

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